

Crystalline Spinon Basis for RSOS Models

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Abstract

The crystalline spinon basis for the RSOS models associated with \widehat{sl}_2 is studied. This basis gives fermionic type character formulas for the branching coefficients of the coset $(\widehat{sl}_2)_l \times (\widehat{sl}_2)_N / (\widehat{sl}_2)_{l+N}$. In addition the path description of the parafermion characters is found as a limit of the spinon description of the string functions.

0 Introduction

The notion of crystalline spinons was introduced and studied in [20] for the case of the higher spin XXZ model. Employing this notion yields a new parametrization of a base of the integrable highest weight $U_q(\widehat{sl}_2)$ modules which naturally leads to the fermionic type character formulas for these modules proposed in [6]. In this paper we study the crystalline spinon basis for RSOS models associated with \widehat{sl}_2 . The idea investigated here is similar to that in [20]. Let us briefly explain this idea in a way that allows comparison to the case of the XXZ model.

The space of states of the RSOS quantum spin chain is

$$\mathcal{W} = \oplus_p \mathbf{C} p,$$

where the sum is over all level k infinite restricted paths $p = [\cdots p_1, p_0, p_{-1}, \cdots]$ satisfying some boundary conditions. A representation theoretical description of this space in the regime III[1, 8] similar to the description of anti-ferromagnetic XXZ models [7, 12] has been given [13, 9]. We have:

$$\mathcal{H} = \oplus \{ U_q(\widehat{sl}_2) \text{ singlet} \} \subset \oplus V(\xi) \otimes V(\eta) \otimes V(\eta')^{*a} \otimes V(\xi')^{*a},$$

where ξ, ξ' and η, η' are level l and N ($l + N = k$) dominant integral weights, respectively. The creation and annihilation operators are given in terms of vertex operators, and their commutation relations are determined [13]. Together, these suggest the following particle picture of the space of states [13, 2]

$$\mathcal{F} = \oplus_{n=0}^{\infty} \left[\oplus_{p,p'} \mathbf{C}((z_1, \cdots, z_n)) \otimes [p_n, \cdots, p_1] \otimes [p'_n, \cdots, p'_1] \right]^{\text{sym}},$$

where p and p' run over all restricted paths of level l and N , respectively, and sym represents symmetrization w.r.t. the S-matrix.

Our main aim in this paper is to establish rigorously the equality $\mathcal{H} = \mathcal{F}$ at $q = 0$. By considering the set of elements of the form $b \otimes b' \otimes b_{-\eta'} \otimes b_{-\xi'}$ in $\mathcal{H}|_{q=0}$, we obtain a crystalline spinon description of the highest weight elements in the tensor products of two integrable highest weight $U_q(\widehat{sl_2})$ modules, where $b_{-\eta'}$ and $b_{-\xi'}$ are the lowest weight elements with lowest weights $-\eta'$ and $-\xi'$ respectively. As a corollary of this parametrization, we obtain the fermionic type formula for the branching coefficients.

In the course of examining the character formulas obtained from spinon descriptions in [20] and in this paper, we have found a path description of the character of the parafermion space. Interestingly, this path description is nothing but the one dimensional configuration sum of the ABF model in regime II in the thermodynamic limit. The evaluation of this sum is the most complicated computation in [1].

The present paper is organized in the following manner. In section 1 we review the RSOS models in the framework of [13, 9]. The crystalline spinon basis for XXZ models is reviewed in section 2. In section 3 we introduce the crystalline creation algebra for the RSOS models. The crystalline spinon description for the space of states of the RSOS models is given in section 4. In section 5 we explain how the spinon formulas of the string functions naturally lead to a path description of the parafermion characters. We briefly explain the derivation of the crystalline creation algebra in Appendix A.

1 Review of the RSOS model in the representation theoretical formulation

Here we review the formulation of the RSOS model given by Jimbo-Miwa-Ohta [13] and introduce notation. We adopt the notation of [13] unless otherwise stated. Let us fix an integer k satisfying $1 \leq N \leq k-1$ and set $l = N - k$. The spin variable of the RSOS model takes the level k dominant integral weights. The Boltzmann weight is given by

$$\begin{array}{c} \lambda \quad - \quad \mu \\ | \quad \quad | \\ \mu' \quad - \quad \nu \end{array} = W_k^N \left(\begin{array}{c} \lambda \quad \mu \\ \mu' \quad \nu \end{array} \middle| z \right),$$

where $W_k^N \left(\begin{array}{c} \lambda \quad \mu \\ \mu' \quad \nu \end{array} \middle| z \right)$ is determined from the commutation relations of vertex operators,

$$\check{R}_{NN} \left(\frac{z_1}{z_2} \right) \Phi_\mu^{\nu V_1^{(N)}}(z_1) \Phi_\lambda^{\mu V_2^{(N)}}(z_2) = \sum_{\mu'} \Phi_\mu^{\nu V_2^{(N)}}(z_2) \Phi_\lambda^{\mu V_1^{(N)}}(z_1) W_k^N \left(\begin{array}{c} \lambda \quad \mu \\ \mu' \quad \nu \end{array} \middle| \frac{z_1}{z_2} \right).$$

Here the Boltzmann weight is zero unless the pairs $(\lambda, \mu), (\mu, \nu), (\nu, \mu'), (\mu', \lambda)$ are admissible. The admissibility condition is specified by the existence condition of the vertex operators. Let us denote by $\lambda_j^{(k)} = (k-j)\Lambda_0 + j\Lambda_1$ the level k dominant integral weight and set $P_k^0 = \{\lambda_j^{(k)} | 0 \leq j \leq k\}$.

Definition 1 *The pair $(\lambda_j^{(k)}, \lambda_{j'}^{(k)})$ is called admissible if the following conditions are satisfied*

$$j - j' \in \{N, N-2, \dots, -N\}, \quad N \leq j + j' \leq 2k - N.$$

Let us define the bijection σ of P_N^0 by $B(\eta) \otimes B^{(N)} \simeq B(\sigma(\eta))$. Explicitly, if $\eta = \lambda_j^{(N)}$ then $\sigma(\eta) = \lambda_{N-j}^{(N)}$. The admissible pairs are parametrized as described below.

Proposition 1 *There is a bijection*

$$P_l^0 \times P_N^0 \simeq \{\text{the admissible pairs}\}$$

given by

$$(\xi, \eta) \mapsto (\xi + \eta, \xi + \sigma(\eta)).$$

Take any $(\xi, \eta), (\tilde{\xi}, \tilde{\eta}) \in P_l^0 \times P_N^0$. Then we state

Definition 2 *$a = (a(n))_{n \in \mathbf{Z}}$ is called a $(\xi\eta, \tilde{\xi}\tilde{\eta})$ restricted path if*

- (i) $a(n) \in P_k^0$, and $(a(n), a(n+1))$ is admissible for any n .
- (ii) $a(n) = \xi + \sigma^n(\eta) \quad (n \gg 0)$, and $a(n) = \tilde{\xi} + \sigma^n(\tilde{\eta}) \quad (n \ll 0)$.

Let us set

$$\begin{aligned} B(\xi, \eta | \tilde{\xi}, \tilde{\eta}) &= B(\xi) \otimes B(\eta) \otimes B(\tilde{\eta})^* \otimes B(\tilde{\xi})^*, \\ B_{\xi\eta, \tilde{\xi}\tilde{\eta}} &= \{b \in B(\xi, \eta | \tilde{\xi}, \tilde{\eta}) | \tilde{e}_i b = \tilde{f}_i b = 0 \text{ for all } i\}, \\ B_{\xi\eta}^\lambda &= \{b \in B(\xi) \otimes B(\eta) | \tilde{e}_i b = 0 \text{ for all } i \text{ and } wt b = \lambda\}, \\ B_{\tilde{\xi}\tilde{\eta}}^* &= \{b \in B(\tilde{\eta})^* \otimes B(\tilde{\xi})^* | \tilde{f}_i b = 0 \text{ for all } i \text{ and } wt b = -\lambda\}. \end{aligned}$$

If we introduce the trivial action of \tilde{e}_i and \tilde{f}_i on $B_{\xi\eta}^\lambda$ and $B_{\tilde{\xi}\tilde{\eta}}^{*\lambda}$, we have the isomorphisms of affine crystals,

$$\begin{aligned} B(\xi) \otimes B(\eta) &\simeq \sqcup_{\lambda \in P_k^0} B_{\xi\eta}^\lambda \otimes B(\lambda), \quad B(\tilde{\eta})^* \otimes B(\tilde{\xi})^* \simeq \sqcup_{\lambda \in P_k^0} B(\lambda)^* \otimes B_{\tilde{\xi}\tilde{\eta}}^{*\lambda}, \\ B_{\xi\eta, \tilde{\xi}\tilde{\eta}} &= \sqcup_{\lambda \in P_k^0} B_{\xi\eta}^\lambda \otimes B_{\tilde{\xi}\tilde{\eta}}^{*\lambda}. \end{aligned}$$

Then

Proposition 2 [9] *There is a bijection*

$$B_{\xi\eta,\tilde{\xi}\tilde{\eta}} \simeq \{(\xi\eta, \tilde{\xi}\tilde{\eta}) \text{ restricted paths}\},$$

given by

$$\begin{aligned} b_\xi \otimes b \otimes b_{-\tilde{\xi}} &\mapsto (a(n))_{n \in \mathbf{Z}}, \\ a(n-1) - a(n) &= \text{wt } p(n) \text{ for any } n, \end{aligned}$$

where $b = (p(n))_{n \in \mathbf{Z}} \in B(\eta) \otimes B(\tilde{\eta})^*$.

In the proof of this proposition, we use the weight multiplicity freeness of $B^{(N)}$.

Proposition 2 motivates the following definition of the space of states of the RSOS quantum spin chain,

$$\begin{aligned} \mathcal{H} &= \oplus \mathcal{H}_{\xi\eta,\tilde{\xi}\tilde{\eta}}, \\ \mathcal{H}_{\xi\eta,\tilde{\xi}\tilde{\eta}} &= \{U_q(\widehat{sl_2}) \text{ singlet}\} \subset V(\xi, \eta | \tilde{\xi}, \tilde{\eta}), \quad \text{and} \\ V(\xi, \eta | \tilde{\xi}, \tilde{\eta}) &= V(\xi) \otimes V(\eta) \otimes V(\tilde{\eta})^{*a} \otimes V(\tilde{\xi})^{*a}. \end{aligned}$$

Here the sum is over all $(\xi, \eta), (\tilde{\xi}, \tilde{\eta}) \in P_l^0 \times P_N^0$. The tensor product is considered to be appropriately completed. Then the crystallized space of states of the RSOS model is $\sqcup B_{\xi\eta,\tilde{\xi}\tilde{\eta}}$, the main object of study in this paper.

The creation operator is defined in terms of the spin 1/2 vertex operators[13],

$$\varphi_{\xi,\eta}^{*\xi',\eta'}(z) = \Phi_{V^{(1)}_\eta}^{\eta'}(z) \Phi_\xi^{\xi' V^{(1)}}(z) \quad (1)$$

which act as

$$\begin{aligned} V(\xi, \eta | \tilde{\xi}, \tilde{\eta}) &\xrightarrow{\Phi_\xi^{\xi' V^{(1)}}(z)} V(\xi') \otimes V_z^{(1)} \otimes V(\eta) \otimes V(\tilde{\eta})^{*a} \otimes V(\tilde{\xi})^{*a} \\ &\xrightarrow{\Phi_{V^{(1)}_\eta}^{\eta'}(z)} V(\xi', \eta' | \tilde{\xi}, \tilde{\eta}). \end{aligned}$$

This operator obviously preserve the space \mathcal{H} , since it is an intertwiner.

2 Crystalline spinon basis for higher spin XXZ models

In this section we recall the results of [20] in a slightly generalized form which will be needed for the case of the RSOS models. We denote by $\mathcal{P}_{\text{res},n}^k(r_1, r_2)$ the set of restricted paths from r_1 to r_2 of length n in the sense of [20]. It is understood that $\mathcal{P}_{\text{res},0}^k = \{\phi\}$. We use the expression $B^{\text{XXZ}}(p_n, \dots, p_1)$ to represent $B(p_n, \dots, p_1)$ of section 2, [20]. Let us set

$$B_{\geq m}^{\text{XXZ}}(p_n, \dots, p_1) = \{\varphi_{j_n}^{*p_n} \dots \varphi_{j_1}^{*p_1} \in B^{\text{XXZ}}(p_n, \dots, p_1) \mid j_1 \geq m\}.$$

The following theorem is proved in [20].

Theorem 1 *Let k be a positive integer and $0 \leq l, r \leq k$. Then there is an isomorphism of affine crystals*

$$\sqcup_{n=0}^{\infty} \sqcup_{(p_n, \dots, p_1) \in \mathcal{P}_{\text{res}, n}^k(r, l)} B^{\text{XXZ}}(p_n, \dots, p_1) \simeq B(\lambda_l) \otimes B(\lambda_r)^*$$

given by

$$\begin{aligned} 1 &\longmapsto [[\]]_r = b_{\lambda_r} \otimes b_{-\lambda_r}, \\ \varphi_{j_n}^{*p_n} \dots \varphi_{j_1}^{*p_1} &\longmapsto [[j_n - p_n, \dots, j_1 - p_1]]. \end{aligned}$$

Although, in [20], the statement of this theorem is only made for the case $r = 0$, the statement given above is actually proved. In fact the bijectivity of the map is obvious, and the condition $r = 0$ is not used in the proof of the weight preservation.

Corollary 1 *The map in Theorem 1 induces the bijection preserving the affine weights:*

$$\sqcup_{n=0}^{\infty} \sqcup_{(p_n, \dots, p_1) \in \mathcal{P}_{\text{res}, n}^k(r, l)} B_{\geq p_1}^{\text{XXZ}}(p_n, \dots, p_1) \simeq B(\lambda_l),$$

where we make the identification:

$$\begin{aligned} B(\lambda_l) &\simeq B(\lambda_l) \otimes b_{-\lambda_r} \\ b &\longrightarrow b \otimes b_{-\lambda_r}. \end{aligned}$$

By a calculation similar to that in section 5 of [20] we have

Corollary 2 *For any $0 \leq r \leq k$, the character $\text{ch}_j(q, z) \equiv \text{tr}_{V(\lambda_j)}(q^{-d} z^{h_1})$ is given by*

$$\text{ch}_j(q, z) = \sum_{n=0}^{\infty} \sum_{m=0}^n \frac{1}{(q)_{n-m}(q)_m} \sum_p q^{h'(p) + mp_1} z^{n+r-2m}, \quad (2)$$

where $p = (p_n, \dots, p_1)$ runs over all level k restricted fusion paths from r to j and $h'(p) = \sum_{s=1}^{n-1} (n-s)H(p_{s+1}, p_s)$.

This corollary gives $k+1$ different expressions for the character of $V(\lambda_j)$.

3 Crystalline creation algebra

In this section we shall introduce the algebra of creation operators of the RSOS model at $q = 0$. Unless otherwise stated we use the notation of [20]. For $p = 0, 1$ let us set $c(p) = 1 - p$.

Definition 3 The crystalline creation algebra $\mathcal{A}^{\text{RSOS}}$ is the associative algebra with unity generated by $\{\varphi_{2j+c(p)}^{*pp'} \mid j \in \mathbf{Z}, p \in \{0,1\}\} \cup \{1\}$ over \mathbf{Z} subject to the following relations:

$$\varphi_{2j_2+c(p_2)}^{*p_2p'_2} \varphi_{2j_1+c(p_1)}^{*p_1p'_1} + \varphi_{2j_1+2s_{21}+c(p_2)}^{*p_2p'_2} \varphi_{2j_2-2s_{21}+c(p_1)}^{*p_1p'_1} = 0, \quad (3)$$

where

$$s_{21} = -1 + H(p_2, p_1) + H(p'_2, p'_1).$$

The algebra $\mathcal{A}^{\text{RSOS}}$ is naturally graded by

$$\begin{aligned} \mathcal{A}^{\text{RSOS}} &= \bigoplus_{n=0}^{\infty} \mathcal{A}_n^{\text{RSOS}} \\ \mathcal{A}_n^{\text{RSOS}} &= \sum \mathbf{Z} \varphi_{2j_n+c(p_n)}^{*p_n p'_n} \cdots \varphi_{2j_1+c(p_1)}^{*p_1 p'_1}, \quad \mathcal{A}_0^{\text{RSOS}} = \mathbf{Z}. \end{aligned}$$

In the following, we denote the crystalline creation algebra of the spin $l/2$ XXZ model[20] by \mathcal{A}^{XXZ} , where the integer $l = k - N$ is associated with the RSOS model as in section 1. For $p = (p_n, \dots, p_1)$ and $p' = (p'_n, \dots, p'_1)$ in $\{0,1\}^n$, let us set

$$\begin{aligned} B^{\text{RSOS}}(p|p') &= \{\varphi_{2j_n+c(p_n)}^{*p_n p'_n} \cdots \varphi_{2j_1+c(p_1)}^{*p_1 p'_1} \mid (j_n, \dots, j_1) \text{ satisfies the condition (4)}\}, \\ B_{\geq 0}^{\text{RSOS}}(p|p') &= \{\varphi_{2j_n+c(p_n)}^{*p_n p'_n} \cdots \varphi_{2j_1+c(p_1)}^{*p_1 p'_1} \in B^{\text{RSOS}}(p|p') \mid j_1 \geq H(p'_1, c(p_1))\}. \end{aligned}$$

If $n = 0$, we define $(p_n, \dots, p_1) = \phi$ and $B^{\text{RSOS}}(\phi|\phi) = \{1\}$. The condition is

$$j_n - I_n - I'_n \geq \cdots \geq j_2 - I_2 - I'_2 \geq j_1, \quad (4)$$

where

$$\begin{aligned} I_m &= I_m(p_m, \dots, p_1) = \sum_{s=1}^{m-1} H(p_{s+1}, p_s), \\ I'_m &= I_m(p'_m, \dots, p'_1) = \sum_{s=1}^{m-1} H(p'_{s+1}, p'_s). \end{aligned}$$

If we set

$$\psi_A(j) = \varphi_{2j+c(p)}^{*pp'}, \quad A = (p, p')$$

and $A_i = (p_i, p'_i)$ ($i = 0, 1$), $s_{A_2 A_1} = s_{21}$, then the commutation relations in Definition 3 can be written as

$$\psi_{A_2}(j_2) \psi_{A_1}(j_1) + \psi_{A_2}(j_1 + s_{A_2 A_1}) \psi_{A_2}(j_2 - s_{A_2 A_1}) = 0.$$

Then the condition (4) is nothing but the normality condition in the sense of Definition 2 in [20]. Hence by Corollary 2 of [20] we have

Theorem 2 $\sqcup_{p,p' \in \{0,1\}^n} B^{\text{RSOS}}(p|p')$ is a \mathbf{Z} linear base of $\mathcal{A}_n^{\text{RSOS}}$.

Definition 4 Let us define the weight of an element of $B^{\text{RSOS}}(p|p')$ by

$$\begin{aligned} \text{wt}(\varphi_{2j_n+c(p_n)}^{*p_n p'_n} \cdots \varphi_{2j_1+c(p_1)}^{*p_1 p'_1}) &= \sum_{s=1}^n \text{wt}(\varphi_{2j_s+c(p_s)}^{*p_s p'_s}) \\ \text{wt}(\varphi_{2j+c(p)}^{*pp'}) &= -j\delta, \\ \text{wt}1 &= 0. \end{aligned}$$

We introduce the structure of a crystal in $B^{\text{RSOS}}(p|p')$ such that \tilde{e}_i, \tilde{f}_i ($i = 0, 1$) act as 0 on any element, and $B^{\text{RSOS}}(p|p')$ has the weights defined above.

Note that the commutation relations (3) are the same as those of $\varphi_{2j_2+c(p_2)}^{*p'_2}$ and $\varphi_{2j_1+c(p_1)}^{*p'_1}$. Hence, by Theorem 1 in [20] and Theorem 2 above, we have

Theorem 3 There is an embedding of the algebra $\mathcal{A}^{\text{RSOS}} \rightarrow \mathcal{A}^{\text{XXZ}}$ given by

$$\varphi_{2j+c(p)}^{*pp'} \mapsto \varphi_{2j+c(p)}^{*p'}.$$

Under this embedding $B^{\text{RSOS}}(p|p')$ is mapped into $B^{\text{XXZ}}(p')$ for $p, p' \in \{0, 1\}^n$. Moreover, the weight in d is preserved under this embedding.

4 Crystalline spinon basis for the RSOS models

In this section we give a new parametrization of a base of the set of highest weight vectors in the tensor products of two integrable highest weight $U_q(\widehat{sl_2})$ modules in terms of crystalline spinons. Let us consider the integers k, N and l as introduced in section 1. We now state the main theorem of this paper.

Theorem 4 For $r_1, l_1 \in \{0, \dots, l\}$ and $r_2, l_2 \in \{0, \dots, N\}$, there is an isomorphism of affine crystals

$$\sqcup_{n=0}^{\infty} \sqcup_{p \in \mathcal{P}_{\text{res}, n}^i(r_1, l_1)} \sqcup_{p' \in \mathcal{P}_{\text{res}, n}^N(r_2, l_2)} B^{\text{RSOS}}(p|p') \simeq B_{\lambda_{l_1}^{(l)} \lambda_{l_2}^{(N)}, \lambda_{r_1}^{(l)} \lambda_{r_2}^{(N)}}$$

given by

$$\begin{aligned} 1 &\longmapsto [[]]_{r_1, r_2} = b_{\lambda_{r_1}^{(l)}} \otimes b_{\lambda_{r_2}^{(N)}} \otimes b_{-\lambda_{r_2}^{(N)}} \otimes b_{-\lambda_{r_1}^{(l)}}, \\ \varphi_{2j_n+c(p_n)}^{*p_n p'_n} \cdots \varphi_{2j_1+c(p_1)}^{*p_1 p'_1} &\longmapsto b_{\lambda_{l_1}^{(l)}} \otimes \varphi_{2j_n+c(p_n)}^{*p'_n} \cdots \varphi_{2j_1+c(p_1)}^{*p'_1} \otimes b_{-\lambda_{r_1}^{(l)}}. \end{aligned}$$

By restricting the above map to the set of elements of the form $b \otimes b' \otimes b_{-\lambda_{r_2}^{(N)}} \otimes b_{-\lambda_{r_1}^{(l)}}$ in $B_{\lambda_{l_1}^{(l)} \lambda_{l_2}^{(N)}, \lambda_{r_1}^{(l)} \lambda_{r_2}^{(N)}}$, we have

Corollary 3 *The map in Theorem 4 induces the bijection preserving the affine weights:*

$$\sqcup_{n=0}^{\infty} \sqcup_{p \in \mathcal{P}_{\text{res},n}^l(r_1, l_1)} \sqcup_{p' \in \mathcal{P}_{\text{res},n}^N(r_2, l_2)} B_{\geq 0}^{\text{RSOS}}(p|p') \simeq B_{\lambda_{l_1}^{(l)} \lambda_{l_2}^{(N)}}^{\lambda_r^{(k)}},$$

where $r = r_1 + r_2$, and we make the identification:

$$\begin{aligned} B_{\lambda_{l_1}^{(l)} \lambda_{l_2}^{(N)}}^{\lambda_r^{(k)}} &\simeq B_{\lambda_{l_1}^{(l)} \lambda_{l_2}^{(N)}}^{\lambda_r^{(k)}} \otimes b_{-\lambda_{r_2}^{(N)}} \otimes b_{-\lambda_{r_1}^{(l)}}, \\ b &\longrightarrow b \otimes b_{-\lambda_{r_2}^{(N)}} \otimes b_{-\lambda_{r_1}^{(l)}}. \end{aligned}$$

Let us define

$$V_{l_1, l_2}^r = \{v \in V(\lambda_{l_1}^{(l)}) \otimes V(\lambda_{l_2}^{(N)}) | e_i v = 0 \ (i = 0, 1), \text{ wt } v = \lambda_r^{(k)}\}.$$

By calculations similar to those in section 5 of [20] we have

Corollary 4 *Assume the same conditions as those in Theorem 4. Then we have*

$$\text{tr}_{V_{l_1, l_2}^r}(q^{-d}) = \sum_{n=0}^{\infty} \frac{1}{(q)_n} \sum_{p \in \mathcal{P}_{\text{res},n}^l(r_1, l_1)} \sum_{p' \in \mathcal{P}_{\text{res},n}^N(r_2, l_2)} q^{h'(p) + h'(p') + nH(p'_1, c(p_1))}.$$

In particular, if $r_1 r_2 = 0$, then

$$\text{tr}_{V_{l_1, l_2}^r}(q^{-d}) = \sum_{n=0}^{\infty} \frac{1}{(q)_n} K_{r_1, l_1}^l(n) K_{r_2, l_2}^N(n),$$

where the polynomial $K_{r,j}^k(n)$ is that defined in (5) of section 5.

We remark that the branching coefficient [14] is obtained from $\text{tr}_{V_{l_1, l_2}^r}(q^{-d})$ as $q^{s_{l_1}^{(l)} + s_{l_2}^{(N)} - s_r^{(k)}} \text{tr}_{V_{l_1, l_2}^r}(q^{-d})$, where $s_m^{(l)} = \frac{(m+1)^2}{4(l+2)} - \frac{1}{8}$.

Example. For the simplest case, $l = N = 1$, one has

$$\begin{aligned} \text{tr}_{V_{0,0}^0}(q^{-d}) &= \sum_{n=0}^{\infty} \frac{q^{2n^2}}{(q)_{2n}}, \\ \text{tr}_{V_{1,1}^0}(q^{-d}) &= \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q)_{2n+1}}, \\ \text{tr}_{V_{0,1}^1}(q^{-d}) &= \sum_{n=0}^{\infty} \frac{q^{n(2n-1)}}{(q)_{2n}} = \sum_{n=0}^{\infty} \frac{q^{n(2n+1)}}{(q)_{2n+1}}. \end{aligned}$$

These are well known fermionic character formulas for the Ising model with $h = 0, 1/2$ and $1/16$ up to the normalization $q^{h-1/48}$ [17].

Theorem 4 follows from Theorem 3 and the following lemma.

Lemma 1 *Let $b = b_{\lambda_{l_1}^{(l)}} \otimes \varphi_{2j_n+c(\epsilon_n)}^{*p'_n} \cdots \varphi_{2j_1+c(\epsilon_1)}^{*p'_1} \otimes b_{-\lambda_{r_1}^{(l)}}$ be an element of $B(\lambda_{l_1}^{(l)}, \lambda_{l_2}^{(N)} | \lambda_{-r_1}^{(l)}, \lambda_{-r_2}^{(N)})$. Then $\tilde{x}_i b = 0$ for $i = 0, 1$ and $x = e, f$ if and only if the path $(\epsilon_n, \dots, \epsilon_1)$ beginning at r_1 is an element of $\mathcal{P}_{\text{res}, n}^N(r_1, l_1)$.*

The following statement is easily proved.

Lemma 2 *Let $b' = b_{c(\mu_n)}^{(1)} \otimes \cdots \otimes b_{c(\mu_1)}^{(1)}$ be an element of $B^{(1)\otimes n}$. Then*

- (i) *If $\tilde{e}_1 b' = \tilde{f}_1 b' = 0$, the path (μ_n, \dots, μ_1) beginning at 0 never falls below 0.*
- (ii) *If $\tilde{e}_0 b' = \tilde{f}_0 b' = 0$, the path (μ_n, \dots, μ_1) beginning at N never rises above N .*

Proof of Lemma 1. Note that the condition $\tilde{e}_i b = \tilde{f}_i b = 0$ for $i = 0, 1$ is equivalent to

$$\begin{aligned} \tilde{x}_1(b_0^{(1)\otimes l_1} \otimes b_{c(\epsilon_n)}^{(1)} \otimes \cdots \otimes b_{c(\epsilon_1)}^{(1)} \otimes b_1^{(1)\otimes r_1}) &= 0 \text{ for } x = e, f, \\ \tilde{x}_0(b_1^{(1)\otimes N-l_1} \otimes b_{c(\epsilon_n)}^{(1)} \otimes \cdots \otimes b_{c(\epsilon_1)}^{(1)} \otimes b_0^{(1)\otimes N-r_1}) &= 0 \text{ for } x = e, f. \end{aligned}$$

Then Lemma 1 follows from Lemma 2. \square .

5 Some other applications

To this point we have shown that the crystalline spinon picture can be naturally extended to the case of RSOS models. In this section, we discuss some additional applications of our construction.

First, we discuss various formulas for the string function of integrable highest weight \widehat{sl}_2 modules which follow from the spinon character formulas of [20]. Then, considering the limiting forms of these formulas, one can obtain the sl_2 parafermion characters for $c = 3k/(k+2) - 1$ as suitable large n limits of the polynomials $K_{r,j}^k(n)$. Finally, we comment on another large n limit of the polynomials $K_{r,j}^k(n)$ and their relation to the Virasoro minimal model characters with $c = 1 - 6/(k+1)(k+2)$.

Let us define $K_{r,j}^k(n)$, ($0 \leq r, j \leq k$).

$$K_{r,j}^k(n) = \sum_{p \in \mathcal{P}_{\text{res}, n}^{(k)}(r, j)} q^{h'(p)}, \quad (5)$$

where $h'(p)$ is given by

$$h'(p) = \sum_{i=1}^{n-1} (n-i)H(p_{i+1}, p_i).$$

By definition, $K_{r,j}^k(n)$ is a polynomial in q with non-negative integer coefficients.

One can prove the following formula

$$\begin{aligned} K_{r,j}^k(n) &= G_{r,j+1}^{k+2}(n) - G_{r,-j-1}^{k+2}(n) \\ G_{r,s}^l(n) &= \sum_{m \in \mathbf{Z}} q^{ms+m^2l} \left[\frac{n}{\frac{s+2ml+n-r-1}{2}} \right]. \end{aligned}$$

The string function¹ $c_\mu^j(q)$ is the character of the weight $h_1 = \mu$ subspace of the integrable module $V(\lambda_j)$. It is easy to read the string functions from the spinon character formula[6, 20],

$$c_\mu^j(q) = q^a \sum_{n=|\mu|}^{\infty} \frac{K_{0,j}^k(n)}{(q)_{\frac{n-\mu}{2}}(q)_{\frac{n+\mu}{2}}},$$

where q^a is a suitable normalization factor depending on μ .

It should be noted that in these formulas the symmetry relations

$$c_\mu^j = c_{-\mu}^j = c_{\mu+2k\mathbf{Z}}^j = c_{k-\mu}^{k-j}$$

are not manifest. Considering this fault as a virtue, one can derive infinitely many q -series identities. For instance, at level $k = 1$, all string functions are equal, and one obtains

$$\sum_{n=0}^{\infty} \frac{q^{n(n+m)}}{(q)_n(q)_{n+m}} = \frac{1}{(q)_\infty},$$

for any $m \geq 0$. The expression on the rhs may be identified with the $m \rightarrow \infty$ limit of the lhs. Similarly, for $k = 2$, one has

$$\begin{aligned} (q)_\infty c_0^0 &= \lim_{n \rightarrow \infty} q^{-2n^2} K_{0,0}^2(4n), \\ (q)_\infty c_2^0 &= \lim_{n \rightarrow \infty} q^{-(2n^2+2n+1)} K_{0,0}^2(4n+2), \\ (q)_\infty c_1^1 &= \lim_{n \rightarrow \infty} q^{-(2n^2+n)} K_{0,1}^2(4n+1). \end{aligned}$$

These are essentially the Virasoro characters for $c = 1/2$ and $h = 0, 1/2$ and $1/16$.

In general, the large n behavior of $K_{r,j}^k(n)$ is described by

¹We normalize it as $c_\mu^\lambda(q) = 1 + \mathcal{O}(q)$.

Lemma 3 For (a) $0 \leq i < k - j$ and (b) $k - j \leq i < k$, put

$$\begin{aligned} (a) \quad & K_{0,j}^k(2nk + 2i + j) = q^{kn^2 + jn + i(2n+1)} \bar{K}_{0,j}^k(2nk + 2i + j), \\ (b) \quad & K_{0,j}^k(2nk + 2i + j) = q^{kn^2 + jn + i(2n+2) + j - k} \bar{K}_{0,j}^k(2nk + 2i + j). \end{aligned}$$

Then, $\bar{K}_{0,j}^k(2nk + 2i + j) = 1 + \mathcal{O}(q)$, and has a limit as $n \rightarrow \infty$ in the form of formal power series in q .

Proof. The lowest degree term comes from the following path

$$\begin{aligned} & (0^k 1^k)^n 0^{i+j} 1^i, \quad (0 \leq i < k - j) \\ & (0^k 1^k)^n 0^k 1^i 0^{i+j-k}, \quad (k - j \leq i < k) \end{aligned}$$

and the number of paths of fixed degree is finite and independent of n (for $n \gg 0$), since those paths can differ from the lowest degree path only near end points. \square

By this lemma, the limiting form of the infinite sequence

$$c_\mu^j = c_{\mu+2k}^j = c_{\mu+4k}^j = \cdots,$$

takes the form

$$c_\mu^j = \frac{1}{(q)_\infty} \lim_{n \rightarrow \infty} \bar{K}_{0,j}^k(2kn + \mu).$$

In particular $\lim_{n \rightarrow \infty} \bar{K}_{0,j}^k(2kn + \mu)$ is an analytical function on $|q| < 1$. Since $(q)_\infty c_\mu^j$ is the parafermion character ² ch_μ^j with $c = 3k/(k+2) - 1$, we have

Proposition 3 The large n limit of the lower degree terms in $K_{0,j}^k(n)$ gives the parafermion character

$$\text{ch}_\mu^j = (q)_\infty c_\mu^j = \lim_{n \rightarrow \infty} \bar{K}_{0,j}^k(2kn + \mu).$$

The sum in $K_{r,j}^k(n)$ looks like the usual path realization of Virasoro minimal model characters [1]. Hence it is natural to seek the relation between these two. In doing this, let us first introduce the 1D configuration sum of the ABF model in regime II or III by

$$X_{j,r}^k(n, q) = \sum_{p \in \mathcal{P}_{res,n}^k(j,r)} q^{\omega'(c(p))},$$

where

$$\omega'(p) = \sum_{i=1}^{n-1} i \tilde{H}(p_{i+1}, p_i).$$

²This is also normalized as $\text{ch}_\mu^j = 1 + \mathcal{O}(q)$.

Here $\tilde{H}(0, 1) = -1$, $\tilde{H}(0, 0) = \tilde{H}(1, 1) = \tilde{H}(1, 0) = 0$, and $c(p) = (c(p_i))$. Let us next set

$$b_{r,s}^j(q) = \text{tr}_{V_{r,s}^j}(q^{-d}),$$

which is the branching coefficient up to the power of q , as mentioned in section 4 for

$$V(\lambda_r^{(k-1)}) \otimes V(\lambda_s^{(1)}) \simeq \oplus_j V_{r,s}^j \otimes V(\lambda_j^{(k)}),$$

where $j \equiv r + s \pmod{2}$. We write $K_{j,r}^k(n, q) = K_{j,r}^k(n)$. Then we have

Proposition 4

(i) $K_{j,r}^k(n, q) = X_{r,j}^k(n, q^{-1})$.

(ii) For $|q| < 1$ we have

$$b_{r,s}^j(q) = \lim_{n \rightarrow \infty} q^{n(n-r)} K_{r+s,j}^k(2n, q^{-1}).$$

Proof. The proof of (i) follows immediately from the definitions. Let us prove (ii). Using the formulation of [9] we have

$$b_{r,s}^j(q) = \lim_{n \rightarrow \infty} \sum_{p \in \mathcal{P}_{\text{res}, 2n}^{(k)}(j, r+s)} q^{\omega'(c(p)) - \omega'(p_{gr}^r)} = \lim_{n \rightarrow \infty} q^{-\omega'(p_{gr}^r)} X_{j, r+s}^k(2n, q),$$

where $p_{gr}^r(n) = \epsilon(n+r)$ and $\epsilon(n) = \frac{1}{2}(1 - (-1)^n)$. Since $\omega'(p_{gr}^r) = -n(n-r)$, (ii) follows from (i). \square

It follows from (i) of this proposition that $\lim_{n \rightarrow \infty} \bar{K}_{0,j}^k(2kn + \mu)$ discussed above is the 1D configuration sum in the thermodynamic limit of the ABF model in regime II.

6 Discussion

In this paper we introduced the crystalline creation algebra for RSOS models. Using this we have given a description of the set of highest weight elements in the tensor product of crystals of integrable highest weight $U_q(\widehat{sl_2})$ modules in terms of crystalline spinons. This crystalline spinon basis leads us to the fermionic type formulas of the branching functions. We have found that the infinite system of string functions obtained from the spinon basis converge to the one dimensional configuration sums (divided by $(q)_\infty$) of the ABF model in regime II. This allows a path description of the parafermion characters.

The fermionic formulas for the branching coefficients associated with the ABF model were first proved in [3]. In that work, however, the relation of the character formula with the underlying quasi-particle structure is rather implicit

or there was no description given of the path counting with weights. Recently O. Warnaar gave another proof by counting weighted paths based on the Fermi gas picture. He also discussed its relation with the Bethe Ansatz solutions [21]. Our results are the rigorous generalization of the character formulas presented in [3, 21] to the case of general RSOS models, directly related with the quasi-particle structure of the model.

The relation between the one dimensional configuration sums of the ABF model in regime II and the parafermion characters found here is consistent with the results of [2]. It is an interesting problem to determine whether this relation can be generalized to the case of general RSOS models and whether there are corresponding spinon type series.

A Appendix

Here we briefly explain the derivation of the commutation relations of the creation operators at $q = 0$ in section 3. In the following calculations, we assume that the creation operators have a well defined $q \rightarrow 0$ limit. We remark that in the main text we do not use this assumption.

We use the notation and results of [13]. The commutation relations of creation operators, as determined in [13], are:

$$\begin{aligned} & \varphi_{\xi'\eta'}^{*\xi''\eta''}(z_1)\varphi_{\xi\eta}^{*\xi'\eta'}(z_2) \\ &= \sum_{\zeta,\kappa} \varphi_{\zeta\kappa}^{*\xi''\eta''}(z_2)\varphi_{\xi\eta}^{*\zeta\kappa}(z_1)W_l^1\left(\begin{smallmatrix} \xi & \xi' \\ \zeta & \xi'' \end{smallmatrix} \middle| \frac{z_1}{z_2}\right)W_N^{*1}\left(\begin{smallmatrix} \eta & \eta' \\ \kappa & \eta'' \end{smallmatrix} \middle| \frac{z_1}{z_2}\right). \end{aligned}$$

As in the case of higher spin XXZ (cf. Appendix A in [20]), the $q \rightarrow 0$ limit of the Boltzmann weights does not depend on the initial weight but on the difference of weights if we remove the fractional powers. So let us set

$$\varphi^{*pp'}(z) = z^{\Delta_\xi - \Delta_{\xi'} + \Delta_\eta - \Delta_{\eta'}} \varphi_{\xi\eta}^{*\xi'\eta'}(z)|_{q=0},$$

where $p, p' = 0, 1$ are defined by

$$\xi' - \xi = (-1)^p(\Lambda_1 - \Lambda_0), \quad \eta' - \eta = (-1)^{p'}(\Lambda_1 - \Lambda_0).$$

Then taking the $q \rightarrow 0$ limit of the commutation relations, we have

$$\varphi^{*p_1p'_1}(z_1)\varphi^{*p_2p'_2}(z_2) = -\left(\frac{z_1}{z_2}\right)^{-1+H(p_1,p_2)+H(p'_1,p'_2)}\varphi^{*p_1p'_1}(z_2)\varphi^{*p_2p'_2}(z_1),$$

where the function H is defined by

$$H(1,0) = 1, \quad H(i,j) = 0 \text{ otherwise.}$$

The algebra in Definition 3 can be obtained by the following mode expansion.

$$\varphi^{*pp'}(z) = \sum_{j \in \mathbf{Z}} \varphi_{2j+c(p)}^{*pp'} z^j.$$

The reason that the mode expansion takes this form is explained as follows. The creation operator is expressed as the composition of type I and type II vertex operators given in (1). Here the type I vertex operator preserves the crystal lattice, and the type II vertex operator is the same as the creation operators of the spin $l/2$ XXZ model. Noting that a highest weight vector in $V(\xi) \otimes V(\eta)$ is dominated in the $q \rightarrow 0$ limit by the element of the form $u_\xi \otimes v$ ($v \in V(\eta)$), we expect a mode expansion as above.

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